STRAIN GRADIENT ELASTICITY THEORY AND ITS APPLICATION TO LATTICE BEAM STRUCTURES

LÝ THUYẾT ĐỘ DỐC BIẾN DẠNG ĐÀN HỒI VÀ ỨNG DỤNG VÀO KẾT CẤU DẦM DẠNG LƯỚI

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Abstract: *This paper investigates the bending behavior of beams by considering: (1) the Euler-Bernoulli beam model, (2) Mindlin's strain gradient elasticity theory, (3) the von Kármán strain assumptions. The principle of virtual work is used to derive the non-linear governing equations in form of a six-order partial differential equation. A conforming Galerkin method based on an isogeometric approach is adopted to naturally fulfill a stringent C^2continuity required by the present beam model. Thereafter, an application to lattice frame structures illustrates the benefits of the present beam model in saving computational costs while maintaining high accuracy as compared to standard 2D finite element simulations*

Keywords: *Strain gradient elasticity, beam model, isogeometric analysis, lattice structure.*

Classification code: *11.2*

Tóm tắt: *Bài báo nghiên cứu ứng xử của dầm có xét đến: (1) lý thuyết dầm Euler-Bernoulli, (2) lý thuyết độ dốc biến dạng đàn hồi của Mindlins, (3) giả thuyết biến dạng von Kármán. Nguyên lý công khả dĩ được sử dụng để xác lập phương trình chủ đạo phi tuyến dưới dạng phương trình vi phân bậc sáu. Tiếp cận Galerkin dựa vào phương phắp đẳng hình học được sử dụng để đáp ứng yêu cầu liên tục nghiêm ngặt C^2 của lý thuyết đề xuất này. Hiệu quả của sử dụng lý thuyết dầm cho kết cấu dạng lưới thể hiện qua việc giảm chi phí tính toán nhưng vẫn đảm bảo tính chính xác cao thông qua so sánh với kết quả mô phỏng dùng phần tử hai chiều.*

Từ khóa: *Đô dốc biến dạng đàn hồi, lý thuyết dầm, đẳng hình học, kết cấu dạng lưới.*

Mã phân loại: *11.2*

1. Introduction

Microbeams are nowadays the key components in micro-and nano- electromechanical systems (MEMS and NEMS, respectively) such as microsensors and actuators, atomic force microscopes, and so on [1[3]. Therefore, they get extreme attention from scientists and researchers as well. As observed through several experimental tests [4[6], the classical continuum mechanics wrongly predicts and describes the static and dynamic behavior of the small-size structures under various mechanical conditions. For example, a decrease in beam/rod size (e.g., thickness or diameter) results in an enhancement of the torsional stiffness of a copper wire [4], a significant increase in the level of plastic hardening of a thin nickel beam [6] or a remarkable increase in the bending rigidity of an epoxy beam [5]. Therefore, it is imperative to develop the non-classical continuum theories which consider additional material length scale parameters besides two Lame's constants used in the conventional elasticity in order to capture the size-effect phenomenon.

Strain gradient elasticity theory pioneered by Mindlin [8] is a well-known non-classical continuum theory that considers higher-order gradient terms in the strain energy density. In the restriction of the current work, we focus only on the strain gradient theory of form II in which the second derivatives of strains are involved. Over the last fifty years, many scientists and researchers have tried to simplify Mindlin's original formulation with fewer additional material parameters as they can, for example, the Aifantis's strain gradient theory [9] (ASGT) involving only one material length scale parameter, for instance, modified strain gradient theory (MSGT) [5] including three material length scale parameters, modified couple stress theory (MCST) [10], simplified strain gradient theory (SSGT) [11].

Furthermore, from the point of view of structural mechanics and experiments, the micro-and nano- beams can exhibit large deformations in which the mid-plane stretching becomes dominant, resulting in geometrical nonlinearity [13]. Within the nonlinear regime, the analytical approaches only work in some simple cases of geometries, loading, and boundary conditions as well. Therefore, numerical techniques are developed for solving complex partial differential equations. Due to the salient features of isogeometric analysis, higher-order continuity achieved by using B-spline shape functions, a conforming isogeometric C^{p-1}-continuous discretization [15] (with order $p\geq 3$) is utilized to naturally fulfill the C^2-continuity requirement without any additional variables.

2. Strain gradient elasticity theory

To capture the size-effect phenomenon, Mindlin [8] considered a higher-order strain gradient tensor $\boldsymbol{\xi}$ which is defined as the derivative of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, it means $\xi_{ijk} = \varepsilon_{jk,i}$, in the strain energy density.

$$U(\boldsymbol{\varepsilon},\boldsymbol{\xi}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\xi_{iik}\xi_{kjj} + a_2\xi_{ijj}\xi_{ikk} + a_3\xi_{iik}\xi_{jjk} + a_4\xi_{ijk}\xi_{ijk} + a_5\xi_{ijk}\xi_{kji} \tag{1}$$

Whereas a_i (i = 1,2,…,5) are non-classical material parameters. With a proper choice of length scale parameters, we can retrieve various some of the one-parameter beam models corresponding to ASGT, MSGT, and MCST or SSGT [12].

3. Geometrically nonlinear Euler–Bernoulli beam model

Let consider a prismatic beam structure with length L, thickness h, and width b. Within the Euler–Bernoulli hypotheses the displacement field is defined as:

$$\begin{cases} u_x(x,y,z) = u(x) - yw'(x) \\ u_y(x,y,z) = w(x) \end{cases} \tag{2}$$

Within the von Kármán strain assumption, the non-zero strain components are defined as:

$$\begin{aligned} \varepsilon_{xx} &= u'_x + \frac{1}{2}\left(u'_y\right)^2 = \varepsilon_0 + y\varepsilon_1 \\ \xi_{xxx} &= \varepsilon_{xx,x} = \left(u'' + w'w''\right) - yw''' \\ \xi_{yxx} &= \varepsilon_{xx,y} = -w'' \end{aligned} \tag{3}$$

Accordingly, the non-zero Cauchy and double stresses for the beam are given as:

$$\begin{aligned} &\sigma_{xx} = E\varepsilon_{xx}, \quad \tau_{xxx} = 2\sum_{I=1}^{5} a_I\xi_{xxx}, \\ &\tau_{yxx} = 2\left(a_2 + a_4\right)\xi_{yxx} \end{aligned} \tag{4}$$

By substituting the definitions of the strain and stress resultants in Eqs. (3) and (4) into Eq.(1), the governing equation can be rewritten in terms of displacements as (5).

Solving a system of the nonlinear equation as in (5) even in the simplest case is a non-trivial task. Therefore, we prefer solving the problem via a weak form equation based on the discrete formulation by using the isogeometric finite element method [15]. Importantly, the nonlinear equation is solved iteratively by the Newton–Raphson scheme, in which the interrupted condition is a difference between two consecutive iterations reduces below a desired error tolerance, e.g., 0.1%. For a detailed description of the solution procedure, one can refer to [16], [17].

4. Numerical examples

4.1 Comparision study

Let us consider a simply supported microbeam with thickness h, length L=20h, and width b=2h subjected to a concentrated load Q=12 mN at the mid-span to make sure that the beam exhibits a relatively large deflection involving geometric nonlinearity. It is assumed that the beam is made of epoxy with material properties as Young's modulus and Poison ratio E = 1.44 GPa and ν = 0.38, respectively, and with the length scale parameter assigned to be equal to l = 17.6 μm[5. Noted that in case of MSGT, the number of length scale parameters reduced to one by setting $l_0 = l_1 = l_2 = l$=17.6 μm. The beam dimension is scaled up, in which the ratio of the thickness to the length scale parameter *h/l* changes in the range of [1, 100]. For the sake of comparison, the results of Dadgar–Rad's work [14] based on MSGT and classical elasticity theories are inserted in Table 1 beside our present solutions based on ASGT, MSGT, MCST, SSGT, and classical elasticity as well within IGA approximation. As noted that, a relation of the material length scale parameter due to particular gradient strain theory is found out from a linear analysis of small-scale beams as follows [12].

$$-EA\left[u'' + w'w''\right] + \alpha_1 A\left(u^{(4)} + 3w''w''' + w'w^{(4)}\right) = f_x$$

$$-EA\left[u'w' + \frac{1}{2}w'^3\right]' + \left(EI + \alpha_2 A\right)w^{(4)} + \alpha_1 A\left[\left(u'''w' + w'^2 w''' + w'w''^2\right)\right]' - \alpha_1 I w^{(6)} = f_y \tag{5}$$

$$\left(l^{MCST}, l^{ASGT}, l^{SSGT}\right) = \left(\sqrt{\frac{53}{15}}, \sqrt{\frac{53(1-2\nu)}{30(1-\nu)}}, \sqrt{\frac{53}{30(1+\nu)}}\right) l^{MSGT} \tag{6}$$

As seen the smallest beam ($h = l$) exhibits the largest deformation with relative deflection $w/h = 2.165$. Thus, the geometrically nonlinear effect becomes significant and makes the beam stiffer. Meanwhile, the largest beam ($h = 100l$) reveals a very small relative deflection $w/h = 0.003$, indicating no geometrically nonlinear effects. Therefore, the transverse displacement coincides with that of linear analysis. This indicates that the present nonlinear finite element formulation works very well for both linear and nonlinear bending analysis. Interestingly, the non-classical theories exhibit higher bending rigidity due to the appearance of gradient terms related to the material length scale parameter *l*. However, the effect is no longer dominant in large-scale structures (e.g., $h/l = 100$).

Let us continue to study the nonlinear behavior of the simply supported microbeams with a given dimension as the thickness of $h = 17.6$ mm and the slenderness of $L/h = 20$ subjected to a uniform distributed load with a magnitude of $q = 30$N/m. Figure 1 plots the load-displacement curves at the mid-span of the beams for different values of length scale parameter *l* scaled-down of [1, 2, 4, 8, 100] times of the thickness *h*. It is observed the linear load-displacement responses in dashed lines are always tangent –at the origin point- to the nonlinear load-displacement curves drawn in solids lines. As seen, an increase in *h/l* ratio essentially increases the beam deflection. As the relative deflection to thickness ratio becomes high (i.e., w/h>1), the geometric nonlinearity comes to play a more essential role through the large difference between the linear and non-linear solutions. Accordingly, the distributions of transverse displacement through the beam coordinate are plotted in figure 2.

Table 1. *Normalized central deflection* $10^3 wEI/(QL^3)$ *of a simply supported micro-beams considering geometrically nonlinear effect under a concentrated load at mid-span Q = 12 mN.*

h/l	**MSGT**		**MCST**		**ASGT**			**SSGT**		**Classical**	
	FEM	**IGA**	***l***	***l* = 33.1**	**FEM**	***l***	***l* = 14.6**	***l***	***l* = 19.9**	**FEM**	**IGA**
1	22.4895	22.4922	32.3925	22.5198	17.8806	17.8823	22.4854	25.1488	22.4854	39.3448	38.1021
2	32.5522	32.5561	40.932	32.5855	28.042	28.0454	32.5488	34.9167	32.5488	45.7811	44.9895
4	40.6884	40.6941	47.0301	40.7175	37.0344	37.0394	40.6883	42.5341	40.6883	50.3008	49.7575
100	4.7275	4.7286	4.7338	4.7286	4.7241	4.7252	4.7285	4.7301	4.7285	4.77309	4.7358

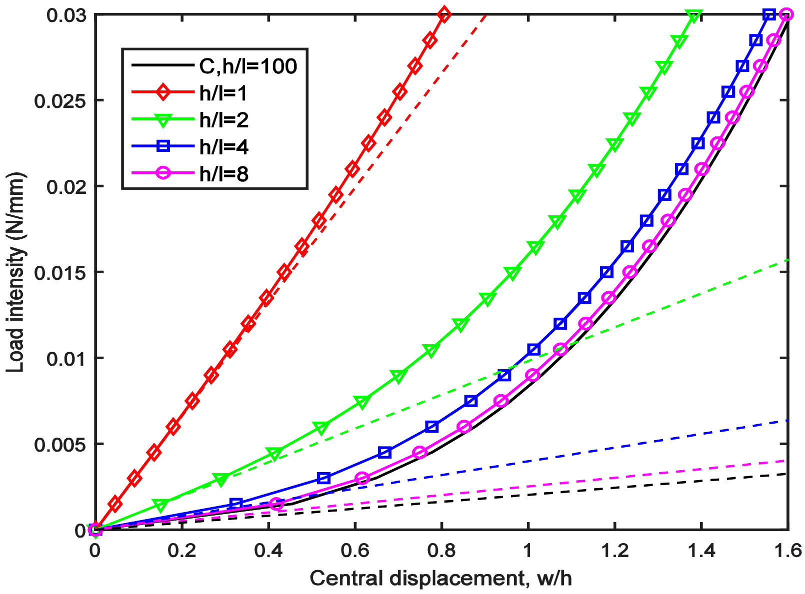

Figure 1. *The load-displacement curve of a SS microbeam.*
Note that linear and nonlinear responses are plotted in dashed and solid lines, respectively.

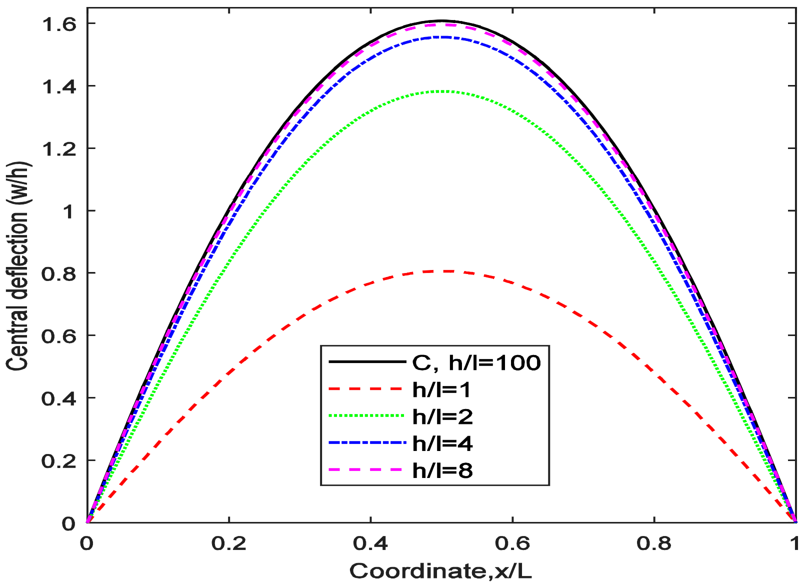

Figure 2*. Deflection of a SS beam under uniform load q=30N/m.*

4.2 Triangular lattice structures

Herein, we further apply the present beam model relying on the strain gradient theory to the real structure-such an elastic triangular lattice frame with dimensions as length L=180 mm, high h = 8.66 mm (Figure 3). The frame is constrained at two ends as a hinge and subjected to a uniformly distributed load, which is applied in increments of Δq = 4N/m until reaches the final magnitude of 200 N/m. As seen, the lattice strip can be produced simply by replicating a unit cell or so-called an RVE shown in figure 3 (lower) (with the dimensions and property from cf. Table 1 in [7]) following to x and y axes. Adopting a homogenization scheme, the frame is homogenized as an isotropic solid beam with equivalent mechanical properties as E_{eff} = 246.7 MPa, ν_{eff} = 0.335, and intrinsic material length scale parameter l = 1.57 mm (based on SSGT). The maximum deflection is recorded at the mid-span and plotted via each increment loading in Figure 4. The red solid line denotes nonlinear solutions while the red dashed line corresponds to the linear ones obtained from COMSOL simulation. As seen, the present results from 1D strain gradient beam model revealed in a red circle and diamond markers corresponding to linear and nonlinear regimes are nearly stood on these red curves. It means that the strain gradient beam model perfectly captures the lattice frame in both linear and nonlinear regimes of deformation.

Next, let us scale up the lattice structure following its dimension by 1, 2, and 4 times according to duplicate the unit cell through the thickness some N = 2, 4, and 8 but still keep certain slenderness of L/h = 20.7. Consequently, the number of DoFs in COMSOL simulation increase in N^2 degrees of freedom (DoFs) resulting in more computational cost as observed in table 2. Meanwhile, the present strain gradient beam model just requires 8 elements associated with 24 DoFs (as using the quartic B-spline basic functions, q = 4) but still achieves a good agreement with COMSOL solutions as shown in figure 5.

Table 2*. A number of DoFs and time consumption.*

N	**COMSOL**		**Beam model**	
	DoFs	**time (s)**	**DoFs**	**time (s)**
2	165366	258	24	0.5
4	651562	880		
8	2586450	4416		

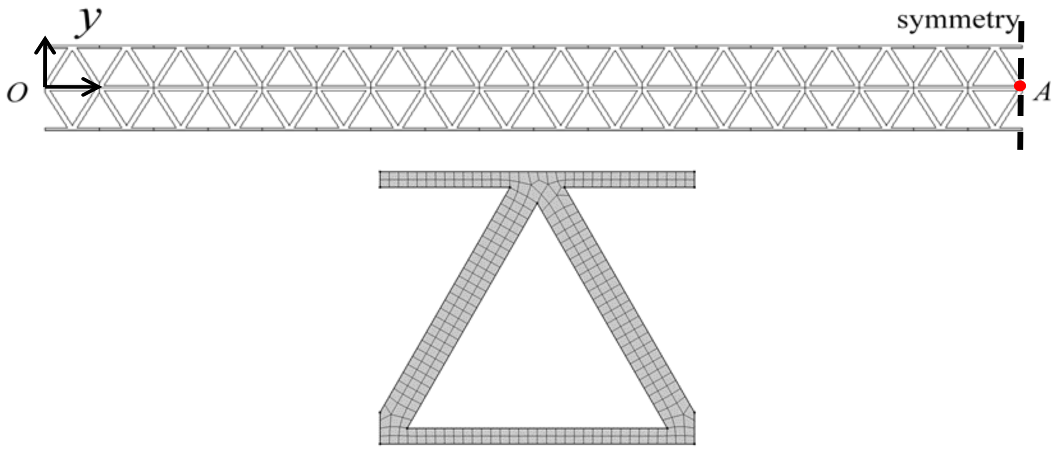

Figure 3*. A haft of lattice frame in a dimension of L=90mm and h=8.66mm is produced by replicating a unit cell 18 and 2 times according to x and y axes, respectively (upper) and (lower) the mesh of a unit cell by COMSOL Multiphysics.*

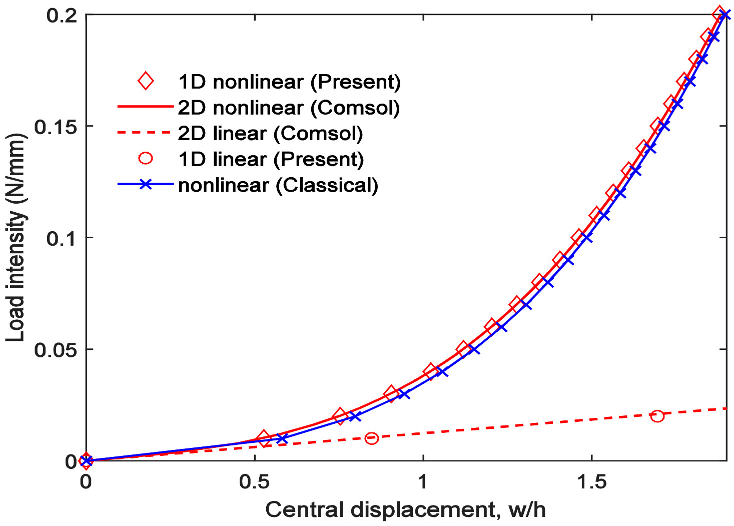

Figure 4*. Load-displacement curve of a lattice beam under uniform load q = 200 N/m, h=8.66 mm, L=180 mm, l=1.57 mm.*

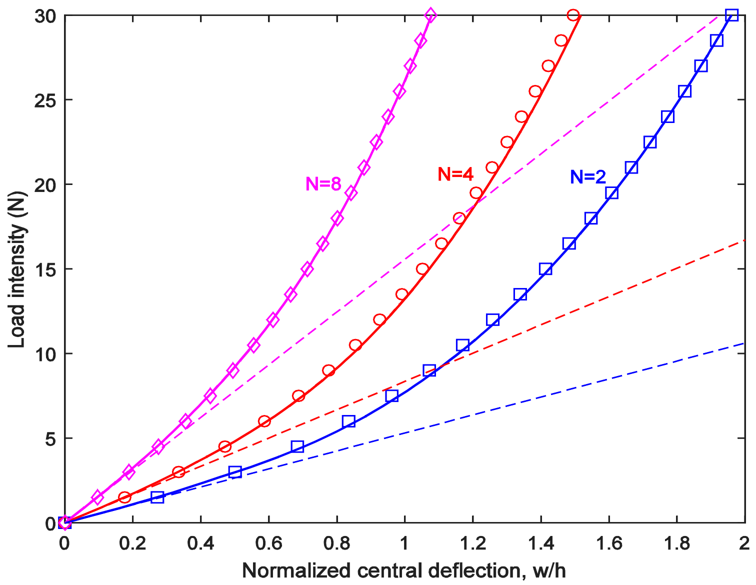

Figure 5*. Load-displacement curves via N-number of unit cell duplicated through the thickness. Linear solutions are depicted in a dashed line while the nonlinear results associated with COMSOL and the strain gradient beam theory are revealed in solid lines and markers, respectively.*

5. Conclusions

This paper studies the nonlinear bending of the E-B beam within Mindlin's strain gradient elasticity theory of form II, which can retrieve some simplified one-parameter strain gradient theories: MSGT, ASGT, MCST, and SSGT. It is concluded that the results are different based on using different simplified theories. By choosing the length scale parameter following Eq. (6), we can maintain almost identical results between them. Also, this relationship works well for both linear and nonlinear regimes of deformation. Furthermore, the geometrical nonlinearity and size effects enable a reduction of the deflection of the beams compared to the classical theory, especially as the material length scale parameter becomes compatible with the beam thickness. However, the nonlinear effect is established from the von Kármán strain assumptions limited to a small or moderate rotation of the cross-section (not exceed 15°). Generally, large deformation of strain gradient elastic beam will be carried out for further works❑

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