

Application of the modified Euler-Lagrange equation to multi-degree-of-freedom problems

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ABSTRACT

In mechanical engineering, systems are often composed of multiple interconnected bodies, leading to complex motions and multiple degrees of freedom. These systems, influenced by various external forces, are challenging to solve, especially when the forces include both conservative and non-conservative elements. Traditional methods, such as the Euler-Lagrange equation and the Lagrange equations of the second kind, are commonly used to derive differential equations for these systems. However, these approaches can be inadequate for systems with intricate motions and mixed types of external forces. To address these limitations, the modified Euler-Lagrange equation offers an effective alternative. This paper presents the theoretical framework of the modified Euler-Lagrange equation and illustrates its application through two detailed examples of multi-degree-of-freedom systems in classical mechanics. Although the paper does not include an in-depth evaluation of the method's effectiveness, it offers a comprehensive approach that simplifies the process of solving a variety of dynamic problems.

1. Introduction

In classical mechanics, the Lagrange equations of the second kind are often used to solve problems involving systems with one or more degrees of freedom. However, for simple systems with a single degree of freedom, applying these equations is less efficient compared to methods like the center of mass motion theorem, the momentum theorem, the angular momentum theorem, the work-energy theorem, or D'Alembert's principle. The

main reason for this inefficiency is the time required to determine the components of the Lagrange equations.

For complex systems with multi-degree-of-freedom, such as those shown in [Figures 1 and 2](#), the Lagrange equations of the second kind are more effective than other methods like the D'Alembert-Lagrange principle [1]. However, when applying the Lagrange equations of the second kind to a system with m degrees of freedom, all components must be calculated, and m differential

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equations must be written. This makes the method less efficient and may be confusing for users. For example, in the mechanical systems shown in Figures 1 and 2, from the 2009 and 2016 National Engineering Mechanics Olympiad (NEMO) exams [2], [3]. These are difficult problems, so the exams are usually designed with systems having two degrees of freedom to simplify manual calculations. However, each system in Figures 1 and 2 actually has three degrees of freedom. The rotation angle φ of the arms CM in Figure 1 and IM in Figure 2 is known, yet the Lagrange equations of the second kind would still require three equations for the three degrees of freedom.

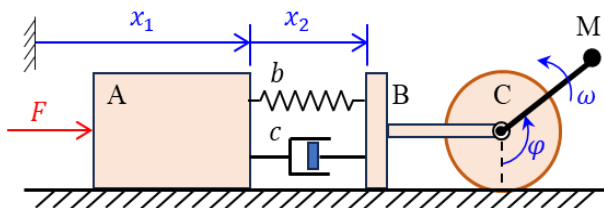


Figure 1. A problem from the 2009 NEMO exam [2].

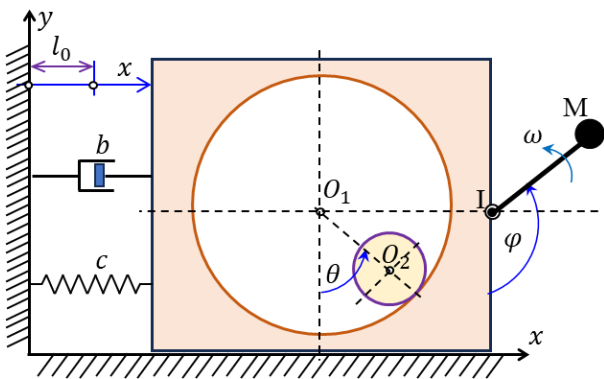


Figure 2. A problem from the 2016 NEMO exam [3].

In cases where the external forces acting on the system are conservative forces, the Euler-Lagrange equation is more effective than the Lagrange equations of the second kind [4], [5], [6], [7]. Looking at the systems shown in Figures 1 and 2, the degree of freedom related to particle M has a known motion pattern and the mass potential energy. The Lagrangian function in the Euler-Lagrange equation already includes

the dynamics of the particle M , so only two differential equations need to be written for the two unknown degrees of freedom. Therefore, using the Euler-Lagrange equation simplifies the calculations compared to using the Lagrange equations of the second kind.

In fact, the external forces acting on the system include both conservative and non-conservative forces (the active force F in Figure 1, the dissipative forces from the dampers in Figures 1 and 2); the Euler-Lagrange equation cannot be used [8]. Therefore, it is necessary to modify the Euler-Lagrange equation to better utilize its advantages, allowing for the solution of multi-degree-of-freedom systems affected by both conservative and non-conservative forces.

This paper introduces the basic theory of the modified Euler-Lagrange equation. In the examples provided, two challenging problems shown in Figures 1 and 2 will be solved in detail using this equation. These examples are presented as originally provided in the MEMO exam problems. The findings of this paper propose a method for solving multi-degree-of-freedom systems compared to the Lagrange equations of the second kind.

2. The modified Euler-Lagrange equation

Consider a mechanical system with n (where $i = 1, 2, \dots, n$) particles. The position of the system is determined by m (where $j = 1, 2, \dots, m$) generalized coordinates (q_1, q_2, \dots, q_m) . Using the vector radius method, the position of the i -th particle is determined by the position vector \mathbf{r}_i given by the following equation:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_m, t) \quad (1)$$

The variation $\delta \mathbf{r}_i$ is expressed in the form of a total differential as follows:

$$\delta \mathbf{r}_i = \sum_{j=1}^m \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (2)$$

The kinetic energy of the system is given by the following equation:

$$T = \frac{1}{2} \sum_{i=1}^n (m_i \dot{\mathbf{r}}_i^2) = \frac{1}{2} \sum_{i=1}^n [(m_i \dot{\mathbf{r}}_i) \dot{\mathbf{r}}_i] \quad (3)$$

The partial derivatives of the kinetic energy T with respect to the generalized coordinate q_j and the generalized velocity \dot{q}_j are given as follows:

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^n \left[(m_i \dot{\mathbf{r}}_i) \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right] \quad (4)$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^n \left[(m_i \dot{\mathbf{r}}_i) \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} \right] \quad (5)$$

According to [9], since q_j and \dot{q}_j are two independent variables, $\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$, they yield:

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^n \left[(m_i \dot{\mathbf{r}}_i) \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \quad (6)$$

The total differential of both sides of equation (6) with respect to time is given as follows:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^n \left[(m_i \ddot{\mathbf{r}}_i) \frac{\partial \mathbf{r}_i}{\partial q_j} \right] + \sum_{i=1}^n \left[(m_i \dot{\mathbf{r}}_i) \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right] \quad (7)$$

According to D'Alembert's principle, $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$, yield:

$$\sum_{i=1}^n \left[(m_i \ddot{\mathbf{r}}_i) \frac{\partial \mathbf{r}_i}{\partial q_j} \right] = \sum_{i=1}^n \left(\mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\delta A}{\delta q_j} = Q_i \quad (8)$$

Where, δA and $\frac{\delta A}{\delta q_j} = Q_i$ represent the virtual work and the generalized force of the forces acting on the system corresponding to the displacement of the generalized

coordinate q_j . Substituting equations (5) and (8) into equation (7), one obtains:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} = Q_i + \frac{\partial T}{\partial q_j} \quad (9)$$

Rearranging equation (9) gives the familiar Lagrange equations of the second kind for the generalized coordinate q_j , as shown in equation (10):

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_i \quad (10)$$

Similarly, the system of equations for m degrees of freedom is formed by the following equation:

$$\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1 \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = Q_2 \\ \dots \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} = Q_m \end{cases} \quad (11)$$

From equation (11), it can be observed that when applying the Lagrange equations of the second kind, a system with m degrees of freedom requires the complete formulation of the corresponding differential equations of motion for the system. In the case where the forces acting on the system are all conservative forces, the motion of the system is conserved. The Lagrange function will be utilized as a substitute for the Lagrange equations of the second kind in equations (10) and (11). Accordingly, the Lagrange function, denoted as L , is defined as follows:

$$L = T(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t) - \Pi(q_1, q_2, \dots) \quad (12)$$

In this context, $T(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t)$ represents the kinetic energy function of the system, while $\Pi(q_1, q_2, \dots)$ denotes the

potential energy function of the conservative forces. For non-conservative forces, $\Pi(q_1, q_2, \dots) = 0$. Equation (12) can be expressed in a simplified form as $L = T - \Pi$; thus, the kinetic energy of the system is determined by the following formula:

$$T = L + \Pi \quad (13)$$

Substituting equation (13) into the left-hand side of equation (10):

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\partial(L + \Pi)}{\partial \dot{q}_j} \right] - \frac{\partial(L + \Pi)}{\partial q_j} \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) + \frac{d}{dt} \left(\frac{\partial \Pi}{\partial \dot{q}_j} \right) - \frac{\partial(L)}{\partial q_j} - \frac{\partial(\Pi)}{\partial q_j} \end{aligned} \quad (14)$$

According to [10], for typical dynamic systems, $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0$, and since the forces acting on the system are conservative forces, it follows that $Q_j = \frac{\partial(\Pi)}{\partial q_j}$. Substituting equation (14) into (10) yields:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial(L)}{\partial q_j} - \frac{\partial(\Pi)}{\partial q_j} = - \frac{\partial(\Pi)}{\partial q_j} \quad (15)$$

So, we get:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial(L)}{\partial q_j} = 0 \quad (16)$$

Equation (16) is referred to as the Euler-Lagrange equation for the j -th degree of freedom. In the case of a system with energy dissipation, this equation becomes:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial(L)}{\partial q_j} = - \frac{\partial D}{\partial \dot{q}_j} \quad (17)$$

Here, D is the energy dissipation function, first introduced by [11]. For a system oscillating close to the equilibrium state, the dissipation function D can be considered as a quadratic form that is non-negative for all

generalized velocities $\dot{\mathbf{q}}$, and the symmetric matrix of damping coefficients c_{jk} is expressed as follows:

$$D(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{j,k=1}^m c_{jk}(\mathbf{q}) \dot{q}_j \dot{q}_k > 0 \quad (18)$$

In the case where the system is subjected to external forces, a generalized force Q_j^* corresponding to each generalized coordinate q_j must be added to the right side of equation (17) [12]; see equation (20). The generalized force Q_j^* is determined from the total virtual work of the external forces acting on the system, corresponding to the differential motion of the generalized coordinate q_j as follows:

$$Q_j^* = \frac{\sum \delta A_k^e}{\delta q_j} \quad (19)$$

Thus, for the j -th degree of freedom, the modified Euler-Lagrange equation is given by equation (20). This equation extends the applicability of the traditional Euler-Lagrange equation, enabling the analysis of mechanical systems subjected to both conservative and non-conservative external forces.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = - \frac{\partial D}{\partial \dot{q}_j} + Q_j^* \quad (20)$$

3. Application examples

3.1. Example 1

The dynamic system under investigation in this work has three degrees of freedom, as seen in Figure 1 [2]. The first degree of freedom corresponds to mass m_1 of rigid body A. It moves in translational motion without friction at the contact surface and is subjected to a constant force $F = \text{const}$. The second degree of freedom involves the pushing arm BC, whose mass is considered negligible. It translates horizontally, driven

by a linear spring with an unstretched length of l and a stiffness constant $c = \text{const}$. The damper device operates linearly, with a constant damping coefficient b , generating a resisting force proportional to the relative velocity between object A and the pushing arm CB. The pushing arm BC is frictionlessly hinged at joint C to a uniform circular disk. The circular disk rolls horizontally without slipping and has a radius of r and a mass of m_2 . The third degree of freedom involves a crank CM that rotates at a constant angular velocity ω around joint C. The crank has a length of e , has negligible mass, and has a mass m attached at the end point M, which is treated as a particle.

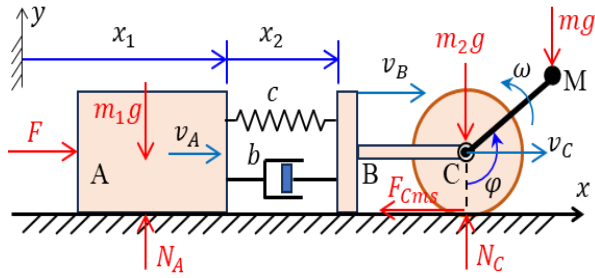


Figure 3. Dynamic analysis diagram for Example 1.

The dynamic analysis diagram of the system is illustrated in Figure 3. The three generalized coordinates corresponding to the problem depicted in Figure 1 are as follows: $q_1 = x_1$, which defines the absolute position of object A; $q_2 = x_2$, which determines the relative position of the pushing arm with respect to object A; and $q_3 = \varphi = \omega t$ (t is the time function), which represents the angle of rotation of the crank CM with respect to the vertical direction. The third degree of freedom in this example is known to be in motion, and the other degrees of freedom are influenced by the crank CM's motion.

The mechanical system shown in Figure 3 is subjected to both potential and non-potential forces. The potential forces are

generated by the spring connecting object A to BC and by the gravitational forces on the objects. The non-potential forces include the external force F and the non-conservative force from the damper device connecting object A to BC. Therefore, the traditional Euler-Lagrange equation cannot solve this problem directly. The solution using the modified Euler-Lagrange equation is presented as follows:

The kinetic energy of object A:

$$\frac{1}{2}m_A v_A^2 = \frac{1}{2}m_1 \dot{x}_1^2 \quad (21)$$

The pushing arm BC has no kinetic energy because it is assumed to have negligible mass. The kinetic energy of the uniform circular disk, which rolls without slipping in a plane, is given by:

$$\begin{aligned} & \frac{1}{2}m_c v_c^2 + \frac{1}{2}J_c \left(\frac{v_c}{r}\right)^2 \\ &= \frac{1}{2}m_2(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2} \frac{m_2 r^2}{2} \left(\frac{\dot{x}_1 + \dot{x}_2}{r}\right)^2 \quad (22) \\ &= \frac{3}{4}m_2(\dot{x}_1 + \dot{x}_2)^2 \end{aligned}$$

The kinetic energy of the particle M is as follows:

$$\frac{1}{2}m v_M^2 \quad (23)$$

The velocity of the particle v_M is obtained by Cartesian coordinate method:

$$v_M^2 = v_{Mx}^2 + v_{My}^2 \quad (24)$$

Where, $x_M = x_1 + x_2 + e \cos\left(\omega t - \frac{\pi}{2}\right)$, and $y_M = e \sin\left(\omega t - \frac{\pi}{2}\right)$. Since x_M and y_M are functions of time, yield: $v_{Mx} = \dot{x}_M = \dot{x}_1 + \dot{x}_2 + e\omega \cos \omega t$, and $v_{My} = \dot{y}_M = e\omega \sin \omega t$.

Therefore, substituting v_{Mx} and v_{My} into equation (24), we obtain:

$$v_M^2 = \dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2 + 2e\omega(\dot{x}_1 + \dot{x}_2)\cos\omega t + (e\omega)^2 \quad (25)$$

Then, substituting the equation (25) into equation (23):

$$T = \frac{1}{2}(m_1 + 1.5m_2 + m)\dot{x}_1^2 + \frac{1}{2}(1.5m_2 + m)\dot{x}_2^2 + (1.5m_2 + m)\dot{x}_1\dot{x}_2 + me\omega(\dot{x}_1 + \dot{x}_2)\cos\omega t + \frac{1}{2}m(e\omega)^2 \quad (27)$$

The potential form of the system is the sum of the mass m , $mge\cos\omega t$; and the spring potential terms, $\frac{1}{2}c(l-x_2)^2$. This can be found by assuming that the potential value is zero attached to the particle M. We get the total of potential as:

$$L = T - \Pi = \frac{1}{2}(m_1 + 1.5m_2 + m)\dot{x}_1^2 + \frac{1}{2}(1.5m_2 + m)\dot{x}_2^2 + (1.5m_2 + m)\dot{x}_1\dot{x}_2 + me\omega(\dot{x}_1 + \dot{x}_2)\cos\omega t + \frac{1}{2}m(e\omega)^2 - \frac{1}{2}c(l-x_2)^2 - mge\cos\omega t \quad (29)$$

Based on equation (20), the partial derivatives of L can be expressed from equation (29) as follows:

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + 1.5m_2 + m)\dot{x}_1 + (1.5m_2 + m)\dot{x}_2 + me\omega\cos\omega t \quad (30)$$

$$\frac{\partial L}{\partial x_1} = 0 \quad (31)$$

$$\frac{\partial L}{\partial x_2} = (1.5m_2 + m)\dot{x}_1 + (1.5m_2 + m)\dot{x}_2 + me\omega\cos\omega t \quad (32)$$

$$\frac{\partial L}{\partial x_2} = c(l-x_2) \quad (33)$$

$$\frac{1}{2}mv_M^2 = \frac{1}{2}m \left[\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2 + 2e\omega(\dot{x}_1 + \dot{x}_2)\cos\omega t + (e\omega)^2 \right] \quad (26)$$

The kinetic energy of the system is obtained by adding up the right-hand sides of equations (21), (22), and (26):

$$\Pi = \frac{1}{2}c(l-x_2)^2 + mge\cos\omega t \quad (28)$$

The Lagrangian function L of the system is defined by equation (12). For the system shown in Figure 3, L is defined as:

Since x_1 and x_2 are time-dependent, the total time derivatives of L are given as follows:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) = (m_1 + 1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 - me\omega^2 \sin\omega t \quad (34)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) = (1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 - me\omega^2 \sin\omega t \quad (35)$$

The generalized forces corresponding to the conservative (external force, F) and non-conservative forces (damping force, $-b\dot{x}_2$) are:

$$Q_{x_1}^* = \frac{F\delta x_1}{\delta x_1} \quad (36)$$

$$Q_{x_2}^* = \frac{(-b\dot{x}_2)\delta x_2}{\delta x_2} = -b\dot{x}_2 \quad (37)$$

By substituting equations (30)-(37) into equation (20), we obtain:

$$(m_1 + 1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 - me\omega^2 \sin \omega t = F \quad (38)$$

$$(1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 - me\omega^2 \sin \omega t = -c(x_2 - l) - b\dot{x}_2 \quad (39)$$

By simplifying, we obtain the system of differential equations of motion, expressed using the modified Euler-Lagrange equation as follows:

$$(m_1 + 1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 = F + me\omega^2 \sin \omega t \quad (40)$$

$$(1.5m_2 + m)\ddot{x}_1 + (1.5m_2 + m)\ddot{x}_2 + b\dot{x}_2 + c(x_2 - l) = me\omega^2 \sin \omega t \quad (41)$$

The system of equations (40) and (41) aligns with the results previously published by [2].

3.2. Example 2

A vibrating grinding table with mass m_1 moves horizontally without friction, as shown in Figure 2 [3]. It is excited by a pendulum mass m (treated as a particle) positioned at a distance e from the rotation axis I, rotating at a constant angular velocity ω . The shock absorber is composed of a linear hydraulic damper with a damping coefficient of b and a spring with a stiffness of c . A uniform circular disk with mass m_2 and radius r rolls without slipping inside a circular hole with radius R in the vibrating grinding table. The generalized coordinates are x and θ . Here, x represents the distance from the free end of the undistorted spring to the edge of the vibrating grinding table, while θ is the angle between the line

through centers O_1O_2 and the vertical direction. The spring has an undistorted length of l_0 . Figure 4 illustrates the dynamic analysis diagram of the system. The mechanical system in Figure 2 has three degrees of freedom. The vibrating grinding table translates horizontally, represented by the generalized coordinate x . The circular disk rolls without slipping within a hole in the grinding table, described by the generalized coordinate θ . The arm IM rotates around joint I at a constant angular velocity ω , with the generalized coordinate $\varphi = \omega t$ already established.

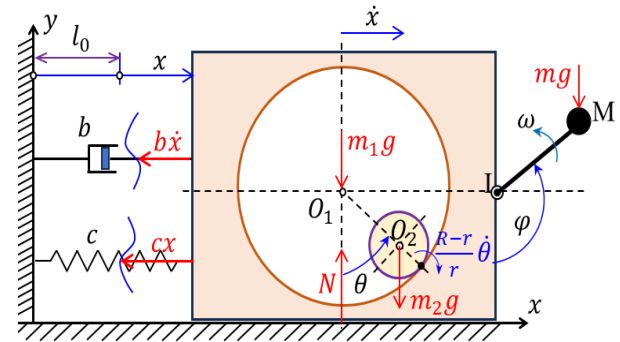


Figure 4. Dynamic analysis diagram for Example 2.

Similar to Example 1, the mechanical system in Figure 4 is subjected to both potential and non-potential forces. The potential forces are generated by the spring in the shock absorber system and the gravitational forces acting on the objects. The non-potential force is the non-conservative force generated by the damper in the shock absorber system. As a result, the problem is solved using the modified Euler-Lagrange equation, as illustrated below:

The kinetic energy of the vibrating grinding table is as follows:

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1\dot{x}^2 \quad (42)$$

The kinetic energy of the uniform circular disk is as follows:

$$\frac{1}{2}m_1v_{O_2}^2 + \frac{1}{2}J_{O_2}\left(\frac{v_{O_2}^r}{r}\right)^2 \quad (43)$$

Where,

$$J_{O_2} = \frac{m_2r^2}{2} \quad (44)$$

$$v_{O_2}^r = (R-r)\dot{\theta} \quad (45)$$

$$v_{O_2}^2 = \dot{x}_2 + (R-r)^2\dot{\theta}^2 + 2\dot{x}(R-r)\dot{\theta}\cos\theta \quad (46)$$

Then, substitute equation (43) into equation (46); we get the kinetic energy of the uniform circular disk is as follows:

$$\frac{1}{2}m_2\left[\dot{x}^2 + \frac{3}{2}(R-r)^2\dot{\theta}^2 + 2(R-r)\dot{x}\dot{\theta}\cos\theta\right] \quad (47)$$

The kinetic energy of the particle M is as follows:

$$\frac{1}{2}mv_M^2 \quad (48)$$

Where,

$$\varphi = \omega t \quad (49)$$

$$\dot{\varphi} = \omega \quad (50)$$

$$v_M^2 = \dot{x}^2 + e^2\omega^2 + 2\dot{x}e\omega\cos\omega t \quad (51)$$

The kinetic energy of the system is obtained by adding up the right-hand sides of equations (42), (47), and (49)-(51):

$$\begin{aligned} T = & \frac{1}{2}m_1\dot{x}^2 \\ & + \frac{1}{2}m_2\left[\dot{x}^2 + \frac{3}{2}(R-r)^2\dot{\theta}^2 \right. \\ & \quad \left. + 2(R-r)\dot{x}\dot{\theta}\cos\theta\right] \\ & + \frac{1}{2}m\left[\dot{x}^2 + e^2\omega^2 + 2\dot{x}e\omega\cos\omega t\right] \end{aligned} \quad (52)$$

We simplify the expression for the kinetic energy T in equation (52), resulting in:

$$\begin{aligned} T = & \frac{1}{2}(m_1 + m_2 + m)\dot{x}^2 + \frac{3}{4}m_2(R-r)^2\dot{\theta}^2 \\ & + m_2(R-r)\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}me^2\omega^2 \\ & + me\omega\dot{x}\cos\omega t \end{aligned} \quad (53)$$

The total potential energy is determined with the potential energy of particle M set as the reference point, making its potential value zero. This energy comprises three components: the gravitational potential energy of mass m_2 , given by $-m_2g(R-r)\cos\theta$; the gravitational potential energy of mass m , represented by $-mge\cos\omega t$; and the spring potential energy term, $\frac{1}{2}cx^2$. Therefore, the total potential energy can be expressed as follows:

$$\Pi = \frac{1}{2}cx^2 - m_2g(R-r)\cos\theta - mge\cos\omega t \quad (54)$$

From equations (53) and (54), the Lagrangian L of the mechanical system shown in Figure 4 is derived as follows:

$$\begin{aligned} L = & T - \Pi \\ = & \frac{1}{2}(m_1 + m_2 + m)\dot{x}^2 + \frac{3}{4}m_2(R-r)^2\dot{\theta}^2 \\ & + m_2(R-r)\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}me^2\omega^2 \\ & + me\omega\dot{x}\cos\omega t - \frac{1}{2}cx^2 \\ & + m_2g(R-r)\cos\theta - mge\cos\omega t \end{aligned} \quad (55)$$

The partial derivatives of L can be derived from equation (29) as follows:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} = & (m_1 + m_2 + m)\dot{x} \\ & + m_2(R-r)\dot{\theta}\cos\theta + me\omega\cos\omega t \end{aligned} \quad (56)$$

$$\frac{\partial L}{\partial x} = -cx \quad (57)$$

$$\frac{\partial L}{\partial \dot{\theta}} = 1.5m_2(R-r)^2 \dot{\theta} + m_2(R-r)\dot{x} \cos \theta \quad (58)$$

$$\frac{\partial L}{\partial \theta} = -m_2(R-r)\dot{x}\dot{\theta} \sin \theta - m_2g(R-r)\sin \theta \quad (59)$$

Since x and θ are functions of time, the total time derivatives of L are given by:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = (m_1 + m_2 + m)\dot{x} + m_2(R-r)(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) - me\omega^2 \sin \omega t \quad (60)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 1.5m_2(R-r)^2 \ddot{\theta} + m_2(R-r)(\ddot{x} \cos \theta - \dot{x}\dot{\theta} \sin \theta) \quad (61)$$

The generalized forces, corresponding to the non-conservative forces generated by the damper in the shock absorber system, are:

$$Q_x^* = \frac{(-b\dot{x})\delta x}{\delta x} = -b\dot{x} \quad (62)$$

$$Q_\theta^* = 0$$

By substituting equations (56)-(62) into equation (20) and simplifying, we obtain the system of differential equations of motion, expressed using the modified Euler-Lagrange equation as follows:

$$(m_1 + m_2 + m)\ddot{x} + m_2(R-r)(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) + b\dot{x} + cx = me\omega^2 \sin \omega t \quad (63)$$

$$\ddot{x} \cos \theta + 1.5(R-r)\ddot{\theta} + g \sin \theta = 0 \quad (64)$$

The system of equations (63) and (64) aligns with the results previously published by [3].

4. Conclusions

This paper presents a modified Euler-Lagrange equation designed to solve multi-degree-of-freedom system problems. It expands the Euler-Lagrange equation's applicability to systems subjected to both conservative and non-conservative forces, simplifying the calculation process compared to the Lagrange equations of the second kind. The paper details the application of the modified Euler-Lagrange equation to multi-degree-of-freedom systems through two examples. To simplify calculations and remain consistent with the MEMO exams, these examples neglect the friction forces at the contact surfaces. This method offers an efficient alternative for solving complex dynamic problems.

Declaration of competing interest and dedication to copyright

The author declares the absence of any potential conflicts of interest from this study and affirms that the paper has not been previously published.

Data available

Data will be provided upon request.

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